

On Free Pseudo-Product Fundamental Graded Lie Algebras

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Abstract. In this paper we first state the classification of the prolongations of complex free fundamental graded Lie algebras. Next we introduce the notion of free pseudo-product fundamental graded Lie algebras and study the prolongations of complex free pseudo-product fundamental graded Lie algebras. Furthermore we investigate the automorphism group of the prolongation of complex free pseudo-product fundamental graded Lie algebras.

Key words: fundamental graded Lie algebra; prolongation; pseudo-product graded Lie algebra

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1 Introduction

Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a graded Lie algebra over the field \mathbb{R} of real numbers or the field \mathbb{C} of complex numbers, and let μ be a positive integer. The graded Lie algebra $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is called a fundamental graded Lie algebra if the following conditions hold: (i) \mathfrak{m} is finite-dimensional; (ii) $\mathfrak{g}_{-1} \neq \{0\}$, and \mathfrak{m} is generated by \mathfrak{g}_{-1} . Moreover a fundamental graded Lie algebra $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is said to be of the μ -th kind if $\mathfrak{g}_{-\mu} \neq \{0\}$, and $\mathfrak{g}_p = \{0\}$ for all $p < -\mu$. It is shown that every fundamental graded algebra $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is prolonged to a graded Lie algebra $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ satisfying the following conditions: (i) $\mathfrak{g}(\mathfrak{m})_p = \mathfrak{g}_p$ for all $p < 0$; (ii) for $X \in \mathfrak{g}(\mathfrak{m})_p$ ($p \geq 0$), $[X, \mathfrak{m}] = \{0\}$ implies $X = 0$; (iii) $\mathfrak{g}(\mathfrak{m})$ is maximum among graded Lie algebras satisfying conditions (i) and (ii) above. The graded Lie algebra $\mathfrak{g}(\mathfrak{m})$ is called the prolongation of \mathfrak{m} . Note that $\mathfrak{g}(\mathfrak{m})_0$ is the Lie algebra of all the derivations of \mathfrak{m} as a graded Lie algebra.

Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a fundamental graded Lie algebra of the μ -th kind, where $\mu \geq 2$. The fundamental graded Lie algebra \mathfrak{m} is called a free fundamental graded Lie algebra of type (n, μ) if the following universal properties hold:

- (i) $\dim \mathfrak{g}_{-1} = n$;
- (ii) Let $\mathfrak{m}' = \bigoplus_{p < 0} \mathfrak{g}'_p$ be a fundamental graded Lie algebra of the μ -th kind and let φ be a surjective linear mapping of \mathfrak{g}_{-1} onto \mathfrak{g}'_{-1} . Then φ can be extended uniquely to a graded Lie algebra epimorphism of \mathfrak{m} onto \mathfrak{m}' .

In Section 3 we see that a universal fundamental graded Lie algebra $b(V, \mu)$ of the μ -th kind introduced by N. Tanaka [11] becomes a free fundamental graded Lie algebra of type (n, μ) , where $\mu \geq 2$, and V is a vector space such that $\dim V = n \geq 2$.

In [13], B. Warhurst gave the complete list of the prolongations of real free fundamental graded Lie algebras by using a Hall basis of a free Lie algebra. The complex version of his theorem has the completely same form except for the ground number field as follows:

Theorem I. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free fundamental graded Lie algebra of type (n, μ) over \mathbb{C} . Then the prolongation $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ of \mathfrak{m} is one of the following types:

- (a) $(n, \mu) \neq (n, 2) (n \geq 2), (2, 3)$. In this case, $\mathfrak{g}(\mathfrak{m})_1 = \{0\}$.
- (b) $(n, \mu) = (n, 2) (n \geq 3), (2, 3)$. In this case, $\dim \mathfrak{g}(\mathfrak{m}) < \infty$ and $\mathfrak{g}(\mathfrak{m})_1 \neq \{0\}$. Furthermore $\mathfrak{g}(\mathfrak{m})$ is isomorphic to a finite-dimensional simple graded Lie algebra of type $(B_n, \{\alpha_n\})$ ($n \geq 3$) or $(G_2, \{\alpha_1\})$ ($n = 2$) (see [15] or Section 5 for the gradations of finite-dimensional simple graded Lie algebras over \mathbb{C}).
- (c) $(n, \mu) = (2, 2)$. In this case, $\dim \mathfrak{g}(\mathfrak{m}) = \infty$. Furthermore, $\mathfrak{g}(\mathfrak{m})$ is isomorphic to the contact algebra $K(1)$ as a graded Lie algebra.

The first purpose of this paper is to give a proof of Theorem I by using the classification of complex irreducible transitive graded Lie algebras of finite depth (cf. [6]). Note that Warhurst's methods in [13] are available to the proof of Theorem I.

Next we introduce the notion of free pseudo-product fundamental graded Lie algebras. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a fundamental graded Lie algebra, and let \mathfrak{e} and \mathfrak{f} be nonzero subspaces of \mathfrak{g}_{-1} . Then \mathfrak{m} is called a pseudo-product fundamental graded Lie algebra with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$ if the following conditions hold: (i) $\mathfrak{g}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$; (ii) $[\mathfrak{e}, \mathfrak{e}] = [\mathfrak{f}, \mathfrak{f}] = \{0\}$ (cf. [10]).

Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a pseudo-product fundamental graded Lie algebra with a pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$, and let $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ be the prolongation of \mathfrak{m} . Moreover let \mathfrak{g}_0 be the Lie algebra of all the derivations of \mathfrak{m} as a graded Lie algebra preserving \mathfrak{e} and \mathfrak{f} . Also for $p \geq 1$ we set $\mathfrak{g}_p = \{X \in \mathfrak{g}(\mathfrak{m})_p : [X, \mathfrak{g}_k] \subset \mathfrak{g}_{p+k} \text{ for all } k < 0\}$ inductively. Then the direct sum $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ becomes a graded subalgebra of $\mathfrak{g}(\mathfrak{m})$, which is called the prolongation of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$.

Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a pseudo-product fundamental graded Lie algebra of the μ -th kind with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$, where $\mu \geq 2$. The pseudo-product fundamental graded Lie algebra $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is called a free pseudo-product fundamental graded Lie algebra of type (m, n, μ) if the following conditions hold:

- (i) $\dim \mathfrak{e} = m$ and $\dim \mathfrak{f} = n$;
- (ii) Let $\mathfrak{m}' = \bigoplus_{p < 0} \mathfrak{g}'_p$ be a pseudo-product fundamental graded Lie algebra of the μ -th kind with pseudo-product structure $(\mathfrak{e}', \mathfrak{f}')$ and let φ be a surjective linear mapping of \mathfrak{g}_{-1} onto \mathfrak{g}'_{-1} such that $\varphi(\mathfrak{e}) \subset \mathfrak{e}'$ and $\varphi(\mathfrak{f}) \subset \mathfrak{f}'$. Then φ can be extended uniquely to a graded Lie algebra epimorphism of \mathfrak{m} onto \mathfrak{m}' .

The main purpose of this paper is to prove the following theorem.

Theorem II. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free pseudo-product fundamental graded Lie algebra of type (m, n, μ) with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$ over \mathbb{C} , and let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$. If $\mathfrak{g}_1 \neq \{0\}$, then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is a finite-dimensional simple graded Lie algebra of type $(A_{m+n}, \{\alpha_m, \alpha_{m+1}\})$.

Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the prolongation of a free pseudo-product fundamental graded Lie algebra $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$ over \mathbb{C} . We denote by $\text{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0$ the group of all the automorphisms as a graded Lie algebra preserving \mathfrak{e} and \mathfrak{f} , which is called the automorphism group of the pseudo-product graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$. In Section 9, we show that $\text{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0$ is isomorphic to $GL(\mathfrak{e}) \times GL(\mathfrak{f})$.

Notation and conventions

- (1) From Section 2 to the last section, all vector spaces are considered over the field \mathbb{C} of complex numbers.
- (2) Let V be a vector space and let W_1 and W_2 be subspaces of V . We denote by $W_1 \wedge W_2$ the subspace of $\Lambda^2 V$ spanned by all the elements of the form $w_1 \wedge w_2$ ($w_1 \in W_1, w_2 \in W_2$).
- (3) Graded vector spaces are always \mathbb{Z} -graded. If we write $V = \bigoplus_{p < 0} V_p$, then it is understood that $V_p = \{0\}$ for all $p \geq 0$. Let $V = \bigoplus_{p \in \mathbb{Z}} V_p$ be a graded vector space. We denote by V_- the subspace $V = \bigoplus_{p < 0} V_p$. Also for $k \in \mathbb{Z}$ we denote by $V_{\leq k}$ the subspace $\bigoplus_{p \leq k} V_p$. Let $V = \bigoplus_{p \in \mathbb{Z}} V_p$ and $W = \bigoplus_{p \in \mathbb{Z}} W_p$ be graded vector spaces. For $r \in \mathbb{Z}$, we set

$$\text{Hom}(V, W)_r = \{\varphi \in \text{Hom}(V, W) : \varphi(V_p) \subset W_{p+r} \text{ for all } p \in \mathbb{Z}\}.$$

2 Free fundamental graded Lie algebras

First of all we give several definitions about graded Lie algebras. Let \mathfrak{g} be a Lie algebra. Assume that there is given a family of subspaces $(\mathfrak{g}_p)_{p \in \mathbb{Z}}$ of \mathfrak{g} satisfying the following conditions:

- (i) $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$;
- (ii) $\dim \mathfrak{g}_p < \infty$ for all $p \in \mathbb{Z}$;
- (iii) $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$ for all $p, q \in \mathbb{Z}$.

Under these conditions, we say that $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is a graded Lie algebra (GLA). Moreover we define the notion of homomorphism, isomorphism, monomorphism, epimorphism, subalgebra and ideal for GLAs in an obvious manner.

A GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is called transitive if for $X \in \mathfrak{g}_p$ ($p \geq 0$), $[X, \mathfrak{g}_-] = \{0\}$ implies $X = 0$, where \mathfrak{g}_- is the negative part $\bigoplus_{p < 0} \mathfrak{g}_p$ of \mathfrak{g} . Furthermore a GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is called irreducible if the \mathfrak{g}_0 -module \mathfrak{g}_{-1} is irreducible.

Let μ be a positive integer. A GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is said to be of depth μ if $\mathfrak{g}_{-\mu} \neq \{0\}$ and $\mathfrak{g}_p = \{0\}$ for all $p < -\mu$.

Next we define fundamental GLAs. A GLA $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is called a *fundamental graded Lie algebra* (FGLA) if the following conditions hold:

- (i) $\dim \mathfrak{m} < \infty$;
- (ii) $\mathfrak{g}_{-1} \neq \{0\}$, and \mathfrak{m} is generated by \mathfrak{g}_{-1} , or more precisely $\mathfrak{g}_{p-1} = [\mathfrak{g}_p, \mathfrak{g}_{-1}]$ for all $p < 0$.

If an FGLA $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is of depth μ , then \mathfrak{m} is also said to be of the μ -th kind. Moreover an FGLA $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is called non-degenerate if for $X \in \mathfrak{g}_{-1}$, $[X, \mathfrak{g}_{-1}] = \{0\}$ implies $X = 0$.

Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be an FGLA of the μ -th kind, where $\mu \geq 2$. \mathfrak{m} is called a free fundamental graded Lie algebra of type (n, μ) if the following conditions hold:

- (i) $\dim \mathfrak{g}_{-1} = n$;
- (ii) Let $\mathfrak{m}' = \bigoplus_{p < 0} \mathfrak{g}'_p$ be an FGLA of the μ -th kind and let φ be a surjective linear mapping of \mathfrak{g}_{-1} onto \mathfrak{g}'_{-1} . Then φ can be extended uniquely to a GLA epimorphism of \mathfrak{m} onto \mathfrak{m}' .

Proposition 2.1. *Let n and μ be positive integers such that $n, \mu \geq 2$.*

- (1) *There exists a unique free FGLA of type (n, μ) up to isomorphism.*
- (2) *Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free FGLA of type (n, μ) . We denote by $\text{Der}(\mathfrak{m})_0$ the Lie algebra of all the derivations of \mathfrak{m} preserving the gradation of \mathfrak{m} . Then the mapping $\Phi : \text{Der}(\mathfrak{m})_0 \ni D \mapsto D|_{\mathfrak{g}_{-1}} \in \mathfrak{gl}(\mathfrak{g}_{-1})$ is a Lie algebra isomorphism.*

Proof. (1) The uniqueness of a free FGLA of type (n, μ) follows from the definition. We set $X = \{1, \dots, n\}$. Let $L(X)$ be the free Lie algebra on X (see [1, Chapter II, § 2]) and let $i : X \rightarrow L(X)$ be the canonical injection. We define a mapping ϕ of X into \mathbb{Z} by $\phi(k) = -1$ ($k \in X$). The mapping ϕ defines the natural gradation $(L(X)_p)_{p < 0}$ on $L(X)$ such that: (i) $L(X)$ is generated by $L(X)_{-1}$; (ii) $\{i(1), \dots, i(n)\}$ is a basis of $L(X)_{-1}$ (see [1, Chapter II, § 2, no. 6]). Note that if $n > 1$, then $L(X)_p \neq 0$ for all $p < 0$. We set $\mathfrak{a} = \bigoplus_{p < -\mu} L(X)_p$; then \mathfrak{a} is a graded ideal of $L(X)$ and the factor GLA $\mathfrak{m} = L(X)/\mathfrak{a}$ becomes an FGLA of the μ -th kind. We put $\mathfrak{a}_p = \mathfrak{a} \cap L(X)_p$ and $\mathfrak{g}_p = L(X)_p/\mathfrak{a}_p$.

Now we prove that $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is a free FGLA of type (n, μ) . Let $\mathfrak{m}' = \bigoplus_{p < 0} \mathfrak{g}'_p$ be an FGLA of the μ -th kind and let φ be a surjective linear mapping of \mathfrak{g}_{-1} onto \mathfrak{g}'_{-1} . Let h be a mapping of X into \mathfrak{m}' defined by $h(k) = \varphi(i(k))$ ($k \in X$). Then there exists a Lie algebra homomorphism \tilde{h} of $L(X)$ into \mathfrak{m}' such that $\tilde{h} \circ i = h$. Since $L(X)$ (resp. \mathfrak{m}') is generated by $L(X)_{-1}$ (resp. \mathfrak{g}'_{-1}), \tilde{h} is surjective. Since $\mathfrak{m}' = \bigoplus_{p < 0} \mathfrak{g}'_p$ is of the μ -th kind, $\tilde{h}(\mathfrak{a}) = 0$, so \tilde{h} induces a GLA epimorphism $L(\varphi)$ of \mathfrak{m} onto \mathfrak{m}' such that $L(\varphi)|_{\mathfrak{g}_{-1}} = \varphi$. The homomorphism $L(\varphi)$ is unique, because $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is generated by \mathfrak{g}_{-1} . Thus \mathfrak{m} is a free FGLA of type (n, μ) .

(2) Assume that \mathfrak{m} is a free FGLA constructed in (1). Let ϕ be an endomorphism of \mathfrak{g}_{-1} . By Corollary to Proposition 8 of [1, Chapter II, § 2, no. 8], ϕ can be extended uniquely to a unique derivation D of $L(X)$. Since $D(L(X)_{-1}) = \phi(L(X)_{-1}) = \phi(\mathfrak{g}_{-1}) \subset L(X)_{-1}$, and since $L(X)$ is generated by $L(X)_{-1}$, we see that $D(L(X)_p) \subset L(X)_p$ and $D(\mathfrak{a}) \subset \mathfrak{a}$. Thus there is a derivation of D_ϕ of \mathfrak{m} such that $\pi \circ D = D_\phi \circ \pi$, where π is the natural projection of $L(X)$ onto \mathfrak{m} . The correspondence $\mathfrak{gl}(\mathfrak{g}_{-1}) \ni \phi \mapsto D_\phi \in \text{Der}(\mathfrak{m})_0$ is an injective linear mapping. Hence $\dim \mathfrak{gl}(\mathfrak{g}_{-1}) \leq \dim \text{Der}(\mathfrak{m})_0$. On the other hand, since \mathfrak{m} is generated by \mathfrak{g}_{-1} , the mapping Φ is a Lie algebra monomorphism. Therefore Φ is a Lie algebra isomorphism. ■

Remark 2.1. Let n and μ be positive integers with $n, \mu \geq 2$, and let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free FGLA of type (n, μ) . Furthermore let $\mathfrak{m}' = \bigoplus_{p < 0} \mathfrak{g}'_p$ be an FGLA of the μ -th kind, and let φ be a linear mapping of \mathfrak{g}_{-1} into \mathfrak{g}'_{-1} .

- (1) From the proof of Proposition 2.1, there exists a unique GLA homomorphism $L(\varphi)$ of \mathfrak{m} into \mathfrak{m}' such that $L(\varphi)|_{\mathfrak{g}_{-1}} = \varphi$.

- (2) Let $\mathfrak{m}'' = \bigoplus_{p < 0} \mathfrak{g}_p''$ be an FGLA of the μ -th kind, and let φ' be a linear mapping of \mathfrak{g}'_{-1} into \mathfrak{g}''_{-1} . Assume that $\mathfrak{m}' = \bigoplus_{p < 0} \mathfrak{g}_p'$ is a free FGLA. By the uniqueness of $L(\varphi' \circ \varphi)$, we see that $L(\varphi' \circ \varphi) = L(\varphi') \circ L(\varphi)$.
- (3) Assume that $\mathfrak{m}' = \bigoplus_{p < 0} \mathfrak{g}_p'$ is a free FGLA and φ is injective. By the result of (2), $L(\varphi)$ is a monomorphism.
- (4) Let W be an m -dimensional subspace of \mathfrak{g}_{-1} with $m \geq 2$. By the result of (3), the subalgebra of \mathfrak{m} generated by W is a free FGLA of type (m, μ) .

By Remark 2.1 (4) and [1, Chapter II, § 2, Theorem 1], we get the following lemma.

Lemma 2.1. *Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free FGLA of type (n, μ) with $\mu \geq 3$. If X, Y are linearly independent elements of \mathfrak{g}_{-1} , then*

$$\begin{aligned} \operatorname{ad}(X)^\mu(Y) &= 0, & \operatorname{ad}(X)^{\mu-1}(Y) &\neq 0, \\ \operatorname{ad}(Y) \operatorname{ad}(X)^{\mu-1}(Y) &= 0, & \operatorname{ad}(Y) \operatorname{ad}(X)^{\mu-2}(Y) &\neq 0. \end{aligned}$$

3 Universal fundamental graded Lie algebras

Following N. Tanaka [11], we introduce universal FGLAs of the μ -th kind.

Let V be an n -dimensional vector space. We define vector spaces $b(V)_p$ ($p < 0$) and linear mappings B_p of $\sum_{r+s=p} b(V)_r \wedge b(V)_s$ into $b(V)_p$ ($p \leq -2$) as follows: First of all, we put $b(V)_{-1} = V$ and $b(V)_{-2} = \Lambda^2 V$. Further we define a mapping $B_{-2} : b(V)_{-1} \wedge b(V)_{-1} \rightarrow b(V)_{-2}$ to be the identity mapping. For $k \leq -3$, we define $b(V)_k$ and B_k inductively as follows: We set $b(V)^{(k+1)} = \bigoplus_{p=-1}^{k+1} b(V)_p$ and we define a subspace $c(V)_k$ of $\Lambda^2(b(V)^{(k+1)})$ to be $\sum_{r+s=k} b(V)_r \wedge b(V)_s$. We denote by $A(V)_k$ the subspace of $c(V)_k$ spanned by the elements

$$\mathfrak{S}_{(X,Y,Z)} \sum_{r+s=k} \sum_{u+v=r} B_r(X_u \wedge Y_v) \wedge Z_s, \quad X, Y, Z \in b(V)^{(k+1)},$$

where $\mathfrak{S}_{(X,Y,Z)}$ stands for the cyclic sum with respect to X, Y, Z , and X_u denotes the $b(V)_u$ -component in the decomposition $b(V)^{(k+1)} = \bigoplus_{p=-1}^{k+1} b(V)_p$. Now we define $b(V)_k$ to be the factor space $c(V)_k / A(V)_k$, and B_k to be the projection of $c(V)_k$ onto $b(V)_k$. We put $b(V) = \bigoplus_{p < 0} b(V)_p$ and define a bracket operation $[\cdot, \cdot]$ on $b(V)$ by

$$[X, Y] = \sum_{p \leq -2} \sum_{r+s=p} B_p(X_r \wedge Y_s)$$

for all $X, Y \in b(V)$. Then $b(V) = \bigoplus_{p < 0} b(V)_p$ becomes a GLA generated by $b(V)_{-1}$, and $b(V)_p \neq 0$ for all $p < 0$ if $\dim V > 1$.

Note that $b(V)_{-3}$ is isomorphic to $\Lambda^2(V) \otimes V / \Lambda^3 V$. Let μ be a positive integer. Assume that $\mu \geq 2$ and $\dim V = n \geq 2$. Since $\bigoplus_{p < -\mu} b(V)_p$ is a graded ideal of $b(V)$, we see that the factor space $b(V, \mu) = b(V) / \bigoplus_{p < -\mu} b(V)_p$ becomes an FGLA of μ -th kind, which is called a universal fundamental graded Lie algebra of the μ -th kind. By [11, Proposition 3.2], $b(V, \mu)$ is a free FGLA of type (n, μ) .

4 The prolongations of fundamental graded Lie algebras

Following N. Tanaka [11], we introduce the prolongations of FGLAs. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be an FGLA. A GLA $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ is called the prolongation of \mathfrak{m} if the following conditions hold:

- (i) $\mathfrak{g}(\mathfrak{m})_p = \mathfrak{g}_p$ for all $p < 0$;
- (ii) $\mathfrak{g}(\mathfrak{m})$ is a transitive GLA;
- (iii) If $\mathfrak{h} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{h}_p$ is a GLA satisfying conditions (i) and (ii) above, then $\mathfrak{h} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{h}_p$ can be embedded in $\mathfrak{g}(\mathfrak{m})$ as a GLA.

We construct the prolongation $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ of \mathfrak{m} . We set $\mathfrak{g}(\mathfrak{m})_p = \mathfrak{g}_p$ ($p < 0$). We define subspaces $\mathfrak{g}(\mathfrak{m})_k$ ($k \geq 0$) of $\text{Hom}(\mathfrak{m}, \bigoplus_{p \leq k-1} \mathfrak{g}(\mathfrak{m})_p)_k$ and a bracket operation on $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ inductively. First $\mathfrak{g}(\mathfrak{m})_0$ is defined to be $\text{Der}(\mathfrak{m})_0$ and a bracket operation $[\cdot, \cdot] : \bigoplus_{p \leq 0} \mathfrak{g}(\mathfrak{m})_p \times \bigoplus_{p \leq 0} \mathfrak{g}(\mathfrak{m})_p \rightarrow \bigoplus_{p \leq 0} \mathfrak{g}(\mathfrak{m})_p$ is defined by

$$\begin{aligned} [X, Y] &= -[Y, X] = X(Y), & X \in \mathfrak{g}(\mathfrak{m})_0, & Y \in \mathfrak{m}, \\ [X, Y] &= XY - YX, & X, Y \in \mathfrak{g}(\mathfrak{m})_0. \end{aligned}$$

Next for $k > 0$ we define $\mathfrak{g}(\mathfrak{m})_k$ ($k \geq 1$) inductively as follows:

$$\mathfrak{g}(\mathfrak{m})_k = \left\{ X \in \text{Hom} \left(\mathfrak{m}, \bigoplus_{p \leq k-1} \mathfrak{g}(\mathfrak{m})_p \right) : X([u, v]) = [X(u), v] + [u, X(v)] \text{ for all } u, v \in \mathfrak{m} \right\},$$

where for $X \in \mathfrak{g}(\mathfrak{m})_r$, $u \in \mathfrak{m}$, we set $[X, u] = -[u, X] = X(u)$. Further for $X \in \mathfrak{g}(\mathfrak{m})_k$, $Y \in \mathfrak{g}(\mathfrak{m})_l$ ($k, l \geq 0$), by induction on $k + l \geq 0$, we define $[X, Y] \in \text{Hom}(\mathfrak{m}, \mathfrak{g}(\mathfrak{m}))_{k+l}$ by

$$[X, Y](u) = [X, [Y, u]] - [Y, [X, u]], \quad u \in \mathfrak{m}.$$

It follows easily that $[X, Y] \in \mathfrak{g}(\mathfrak{m})_{k+l}$. With this bracket operation, $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ becomes a graded Lie algebra satisfying conditions (i), (ii) and (iii) above.

Let \mathfrak{m} and $\mathfrak{g}(\mathfrak{m})$ be as above. Assume that we are given a subalgebra \mathfrak{g}_0 of $\mathfrak{g}(\mathfrak{m})_0$. We define subspaces \mathfrak{g}_k ($k \geq 1$) of $\mathfrak{g}(\mathfrak{m})_k$ inductively as follows:

$$\mathfrak{g}_k = \{ X \in \mathfrak{g}(\mathfrak{m})_k : [X, \mathfrak{g}_p] \subset \mathfrak{g}_{p+k} \text{ for all } p < 0 \}.$$

If we put $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$, then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ becomes a transitive graded Lie subalgebra of $\mathfrak{g}(\mathfrak{m})$, which is called the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$.

By Proposition 2.1 (2) we get the following proposition.

Proposition 4.1. *Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free FGLA and let $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ be the prolongation of \mathfrak{m} . Then the mapping $\mathfrak{g}(\mathfrak{m})_0 \ni D \mapsto D|_{\mathfrak{g}_{-1}} \in \mathfrak{gl}(\mathfrak{g}_{-1})$ is an isomorphism.*

Conversely we obtain the following proposition.

Proposition 4.2. *Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be an FGLA of the μ -th kind and let $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ be the prolongation of \mathfrak{m} . Assume that $\mathfrak{g}(\mathfrak{m})_0$ is isomorphic to $\mathfrak{gl}(\mathfrak{g}_{-1})$. If $\mu = 2$ or $\mu = 3$, then \mathfrak{m} is a free FGLA.*

Proof. We put $n = \dim \mathfrak{g}_{-1}$. We consider a universal FGLA $b(\mathfrak{g}_{-1}, \mu) = \bigoplus_{p < 0} b(\mathfrak{g}_{-1}, \mu)_p$ of the μ -th kind. Since $b(\mathfrak{g}_{-1}, \mu)$ is a free FGLA of type (n, μ) , there exists a GLA epimorphism φ of $b(\mathfrak{g}_{-1}, \mu)$ onto \mathfrak{m} such that the restriction $\varphi|_{b(\mathfrak{g}_{-1}, \mu)_{-1}}$ is the identity mapping. Let $\check{b}(\mathfrak{g}_{-1}, \mu) = \bigoplus_{p \in \mathbb{Z}} \check{b}(\mathfrak{g}_{-1}, \mu)_p$ be the prolongation of $b(\mathfrak{g}_{-1}, \mu)$. Since the mapping $\mathfrak{g}(\mathfrak{m})_0 \ni D \mapsto D|_{\mathfrak{g}_{-1}} \in \mathfrak{gl}(\mathfrak{g}_{-1})$ is an isomorphism, φ can be extended to be a homomorphism $\check{\varphi}$ of $\bigoplus_{p \leq 0} \check{b}(\mathfrak{g}_{-1}, \mu)_p$ onto $\bigoplus_{p \leq 0} \mathfrak{g}(\mathfrak{m})_p$. Let \mathfrak{a} be the kernel of $\check{\varphi}$; then \mathfrak{a} is a graded ideal of $\bigoplus_{p \leq 0} \check{b}(\mathfrak{g}_{-1}, \mu)_p$. We set $\mathfrak{a}_p = \mathfrak{a} \cap \check{b}(\mathfrak{g}_{-1}, \mu)_p$; then $\mathfrak{a} = \bigoplus_{p \leq 0} \mathfrak{a}_p$. Since the restriction of $\check{\varphi}$ to $\check{b}(\mathfrak{g}_{-1}, \mu)_{-1} \oplus \check{b}(\mathfrak{g}_{-1}, \mu)_0$ is injective, $\mathfrak{a}_p = \{0\}$ for $p \geq -1$. Also each \mathfrak{a}_p is a $\check{b}(\mathfrak{g}_{-1}, \mu)_0$ -submodule of $\check{b}(\mathfrak{g}_{-1}, \mu)_p$. From the construction of $b(\mathfrak{g}_{-1}, \mu)$, we see that $b(\mathfrak{g}_{-1}, \mu)_{-2}$ (resp. $b(\mathfrak{g}_{-1}, \mu)_{-3}$) is isomorphic to $\Lambda^2(\mathfrak{g}_{-1})$ (resp. $\Lambda^2(\mathfrak{g}_{-1}) \otimes \mathfrak{g}_{-1}/\Lambda^3(\mathfrak{g}_{-1})$) as a $\check{b}(\mathfrak{g}_{-1}, \mu)_0$ -module. By the table of [8], $\Lambda^2(\mathfrak{g}_{-1})$ and $\Lambda^2(\mathfrak{g}_{-1}) \otimes \mathfrak{g}_{-1}/\Lambda^3(\mathfrak{g}_{-1})$ are irreducible $\mathfrak{gl}(\mathfrak{g}_{-1})$ -modules. Thus we see that $\mathfrak{a}_{-2} = \mathfrak{a}_{-3} = \{0\}$. From $\mu \leq 3$ it follows that φ is an isomorphism. ■

5 Finite-dimensional simple graded Lie algebras

Following [15], we first state the classification of finite-dimensional simple GLAs.

Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a finite-dimensional simple GLA of the μ -th kind over \mathbb{C} such that the negative part \mathfrak{g}_- is an FGLA. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g}_0 ; then \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} such that $E \in \mathfrak{h}$, where E is the element of \mathfrak{g}_0 such that $[E, x] = px$ for all $x \in \mathfrak{g}_p$ and p . Let Δ be a root system of $(\mathfrak{g}, \mathfrak{h})$. For $\alpha \in \Delta$, we denote by \mathfrak{g}^α the root space corresponding to α . We set $\mathfrak{h}_{\mathbb{R}} = \{h \in \mathfrak{h} : \alpha(h) \in \mathbb{R} \text{ for all } \alpha \in \Delta\}$ and let (h_1, \dots, h_l) be a basis of $\mathfrak{h}_{\mathbb{R}}$ such that $h_1 = E$. We define the set of positive roots Δ^+ as the set of roots which are positive with respect to the lexicographical ordering in $\mathfrak{h}_{\mathbb{R}}^*$ determined by the basis (h_1, \dots, h_l) of $\mathfrak{h}_{\mathbb{R}}$. Let $\Pi \subset \Delta^+$ be the corresponding simple root system. We denote by $\{m_1, \dots, m_l\}$ the coordinate functions corresponding to Π , i.e., for $\alpha \in \Delta$, we can write $\alpha = \sum_{i=1}^l m_i(\alpha) \alpha_i$.

We set $\alpha_i(E) = s_i$ and $\mathbf{s} = (s_1, \dots, s_l)$; then each s_i is a non-negative integer. For $\alpha \in \Delta$, we call the integer $\ell_{\mathbf{s}}(\alpha) = \sum_{i=1}^l m_i(\alpha) s_i$ the \mathbf{s} -length of α . We put $\Delta_p = \{\alpha \in \Delta : \ell_{\mathbf{s}}(\alpha) = p\}$, $\Pi_p = \Delta_p \cap \Pi$ and $I = \{i \in \{1, \dots, l\} : s_i = 1\}$. Let θ be the highest root of \mathfrak{g} ; then $\ell_{\mathbf{s}}(\theta) = \mu$. Also since the \mathfrak{g}_0 -module $\mathfrak{g}_{-\mu}$ is irreducible, $\dim \mathfrak{g}_{-\mu} = 1$ if and only if $\langle \theta, \alpha_i^\vee \rangle = 0$ for all $i \in \{1, \dots, l\} \setminus I$, where $\{\alpha_i^\vee\}$ is the simple root system of the dual root system Δ^\vee of Δ corresponding to $\{\alpha_i\}$. In our situation, since \mathfrak{g}_- is generated by \mathfrak{g}_{-1} , we have $s_i = 0$ or 1 for all i . The l -tuple $\mathbf{s} = (s_1, \dots, s_l)$ of non-negative integers is determined only by the ordering of $(\alpha_1, \dots, \alpha_l)$. In what follows, we assume that the ordering of $(\alpha_1, \dots, \alpha_l)$ is as in the table of [2]. If \mathfrak{g} has the Dynkin diagram of type X_l ($X = A, \dots, G$), then the simple GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is said to be of type (X_l, Π_1) . Here we remark that for an automorphism $\bar{\mu}$ of the Dynkin diagram, a simple GLA of type (X_l, Π_1) is isomorphic to that of type $(X_l, \bar{\mu}(\Pi_1))$. We will identify a simple GLA of type (X_l, Π_1) with that of type $(X_l, \bar{\mu}(\Pi_1))$.

For $i \in I$, we put $\Delta_p^{(i)} = \{\alpha \in \Delta : m_i(\alpha) = p \text{ and } m_j(\alpha) = 0 \text{ for } j \in I \setminus \{i\}\}$ and $\mathfrak{g}_p^{(i)} = \sum_{\alpha \in \Delta_p^{(i)}} \mathfrak{g}^\alpha$; then $\mathfrak{g}_{-1}^{(i)}$ is an irreducible \mathfrak{g}_0 -submodule of \mathfrak{g}_{-1} with highest weight $-\alpha_i$. In particular, if the \mathfrak{g}_0 -module \mathfrak{g}_{-1} is irreducible, then $\#(I) = 1$.

For $i \in I$, we denote by $\mathfrak{g}^{(i)}$ the subalgebra of \mathfrak{g} generated by $\mathfrak{g}_{-1}^{(i)} \oplus \mathfrak{g}_1^{(i)}$; then $\mathfrak{g}^{(i)}$ is a simple GLA whose Dynkin diagram is the connected component containing the vertex i of the subdiagram of X_l corresponding to vertices $(\{1, \dots, l\} \setminus I) \cup \{i\}$. We denote by $\theta^{(i)}$ the highest root of $\mathfrak{g}^{(i)}$. Then $[\mathfrak{g}_{-1}^{(i)}, \mathfrak{g}_{-1}^{(i)}] = \{0\}$ if and only if $m_i(\theta^{(i)}) = 1$.

From Theorem 5.2 of [15], we obtain the following theorem:

Theorem 5.1. *Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a finite-dimensional simple GLA over \mathbb{C} such that \mathfrak{g}_- is an FGLA and the \mathfrak{g}_0 -module \mathfrak{g}_{-1} is irreducible. Then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is the prolongation of \mathfrak{g}_- except for the following cases:*

- (a) \mathfrak{g}_- is of the first kind;
- (b) \mathfrak{g}_- is of the second kind and $\dim \mathfrak{g}_{-2} = 1$.

Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a finite-dimensional simple GLA. Now we assume that \mathfrak{g}_0 is isomorphic to $\mathfrak{gl}(\mathfrak{g}_{-1})$; then the \mathfrak{g}_0 -module \mathfrak{g}_{-1} is irreducible. The derived subalgebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ of \mathfrak{g}_0 is a semisimple Lie algebra whose Dynkin diagram is the subdiagram of X_l consisting of the vertices $\{1, \dots, l\} \setminus I$. Since $[\mathfrak{g}_0, \mathfrak{g}_0]$ is of type A_{l-1} and since the \mathfrak{g}_0 -module \mathfrak{g}_{-1} is elementary, (X_l, Δ_1) is one of the following cases:

$$(A_l, \{\alpha_1\}), \quad (B_l, \{\alpha_l\}), \quad l \geq 2, \quad (G_2, \{\alpha_1\}).$$

From this result and Propositions 4.1 and 4.2, we get the following theorem:

Theorem 5.2. *Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a finite-dimensional simple GLA of type (X_l, Π_1) over \mathbb{C} satisfying the following conditions:*

- (i) \mathfrak{g}_- is an FGLA of the μ -th kind;
- (ii) The \mathfrak{g}_0 -module \mathfrak{g}_{-1} is irreducible;
- (iii) \mathfrak{g}_0 is isomorphic to $\mathfrak{gl}(\mathfrak{g}_{-1})$;
- (iv) \mathfrak{g} is the prolongation of \mathfrak{g}_- .

Then \mathfrak{g}_- is a free FGLA of type (l, μ) , and $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is one of the following types:

- (a) $l \geq 3, \mu = 2, (X_l, \Pi_1) = (B_l, \{\alpha_l\})$.
- (b) $l = 2, \mu = 3, (X_l, \Pi_1) = (G_2, \{\alpha_1\})$.

6 Graded Lie algebras $W(n)$, $K(n)$ of Cartan type

In this section, following V.G. Kac [3], we describe Lie algebras $W(n)$, $K(n)$ of Cartan type and their standard gradations.

Let $A(m)$ denote the monoid (under addition) of all m -tuples of non-negative integers. For an m -tuple $\mathbf{s} = (s_1, \dots, s_m)$ of positive integers and $\alpha = (\alpha_1, \dots, \alpha_m) \in A(m)$ we set $\|\alpha\|_{\mathbf{s}} = \sum_{i=1}^m s_i \alpha_i$. Also we denote the m -tuple $(1, \dots, 1)$ by $\mathbf{1}_m$ and we denote the $(m+1)$ -tuple $(1, \dots, 1, 2)$ by $(\mathbf{1}_m, 2)$. Let $\mathfrak{A}(m) = \mathbb{C}[x_1, \dots, x_m]$. For any m -tuple \mathbf{s} of positive integers, we denote by $\mathfrak{A}(m; \mathbf{s})_p$ the subspace of $\mathfrak{A}(m)$ spanned by polynomials

$$x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}, \quad \alpha = (\alpha_1, \dots, \alpha_m) \in A(m), \quad \|\alpha\|_{\mathbf{s}} = p.$$

Let $W(m)$ be the Lie algebra consisting of all the polynomial vector fields

$$\sum_{i=1}^m P_i \frac{\partial}{\partial x_i}, \quad P_i \in \mathfrak{A}(m). \quad (6.1)$$

For an m -tuple $\mathbf{s} = (s_1, \dots, s_m)$ of positive integers, we denote by $W(m; \mathbf{s})_p$ the subspaces of $W(m)$ consisting of those polynomial vector fields (6.1) such that the polynomials P_i are contained in $\mathfrak{A}(m; \mathbf{s})_{p+s_i}$; then $W(m; \mathbf{s}) = \bigoplus_{p \in \mathbb{Z}} W(m; \mathbf{s})_p$ is a transitive GLA. In particular, $W(m; \mathbf{1}_m) = \bigoplus_{p \geq -1} W(m; \mathbf{1}_m)_p$ is a transitive irreducible GLA such that: (i) $W(m; \mathbf{1}_m)_0$ is isomorphic to $\mathfrak{gl}(m, \mathbb{C})$; (ii) the $W(m; \mathbf{1}_m)_0$ -module $W(m; \mathbf{1}_m)_{-1}$ is elementary; (iii) $W(m; \mathbf{1}_m)$ is the prolongation of $W(m; \mathbf{1}_m)_-$.

We now consider the following differential form

$$\omega_K = dx_{2n+1} - \sum_{i=1}^n x_{i+n} dx_i.$$

Define

$$K(n) = \{D \in W(2n+1) : D\omega_K \in \mathfrak{A}(2n+1)\omega_K\}.$$

(Here the action of D on the differential forms is extended from its action $\mathfrak{A}(2n+1)$ by requiring that D be derivation of the exterior algebra satisfying $D(df) = d(Df)$, where $df = \sum \frac{\partial f}{\partial x_i} dx_i$, $f \in \mathfrak{A}(m)$.) We set $K(n)_p = W(2n+1; (\mathbf{1}_{2n}, 2))_p \cap K(n)$. Then $K(n) = \bigoplus_{p \geq -2} K(n)_p$ is a transitive irreducible GLA such that: (i) $K(n)_0$ is isomorphic to $\mathfrak{csp}(n, \mathbb{C})$; (ii) the $K(n)_0$ -module $K(n)_{-1}$ is elementary; (iii) $K(n)$ is the prolongation of $K(n)_-$ (cf. [3, 5]).

From Proposition 2.2 of [6], we get

Theorem 6.1. *Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a transitive GLA over \mathbb{C} satisfying the following conditions:*

- (i) \mathfrak{g}_- is an FGLA of the μ -th kind;
- (ii) \mathfrak{g} is infinite-dimensional;
- (iii) The \mathfrak{g}_0 -module \mathfrak{g}_{-1} is irreducible;
- (iv) \mathfrak{g} is the prolongation of \mathfrak{g}_- .

Then $\mu \leq 2$ and $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is isomorphic to $W(m; \mathbf{1}_m)$ or $K(n)$.

7 Classification of the prolongations of free fundamental graded Lie algebras

Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free FGLA of type (n, μ) over \mathbb{C} , and let $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ be the prolongation of \mathfrak{m} . First of all, we assume that $\dim \mathfrak{g}(\mathfrak{m}) = \infty$. By Theorem 6.1, $\mathfrak{g}(\mathfrak{m})$ is isomorphic to $K(m)$ as a GLA, where $n = 2m$. Since $K(m)_0$ is isomorphic to $\mathfrak{csp}(m, \mathbb{C})$ and since $\mathfrak{g}(\mathfrak{m})_0$ is isomorphic to $\mathfrak{gl}(n, \mathbb{C})$, we see that $m = 1$. Therefore $\mathfrak{g}(\mathfrak{m})$ is isomorphic to $K(1)$ as a GLA.

Next we assume that $\dim \mathfrak{g}(\mathfrak{m}) < \infty$ and $\mathfrak{g}(\mathfrak{m})_1 \neq 0$. Since the $\mathfrak{g}(\mathfrak{m})_0$ -module $\mathfrak{g}(\mathfrak{m})_{-1}$ is irreducible, $\mathfrak{g}(\mathfrak{m})$ is a finite-dimensional simple GLA (see [4, 7]). By Theorem 5.2, $\mathfrak{g}(\mathfrak{m})$ is isomorphic to one of the following types:

$$(B_l, \{\alpha_l\}) \quad l \geq 3, \quad (G_2, \{\alpha_1\}).$$

Thus we get a proof of the following theorem:

Theorem 7.1. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free FGLA of type (n, μ) over \mathbb{C} , and let $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ be the prolongation of \mathfrak{m} . Then one of the following cases occurs:

- (a) $(n, \mu) \neq (n, 2) (n \geq 2), (2, 3)$. In this case, $\mathfrak{g}(\mathfrak{m})_1 = \{0\}$.
- (b) $(n, \mu) = (n, 2) (n \geq 3), (2, 3)$. In this case, $\dim \mathfrak{g}(\mathfrak{m}) < \infty$ and $\mathfrak{g}(\mathfrak{m})_1 \neq \{0\}$. Furthermore $\mathfrak{g}(\mathfrak{m})$ is isomorphic to a finite-dimensional simple GLA of type $(B_n, \{\alpha_n\}) (n \geq 3)$ or $(G_2, \{\alpha_1\}) (n = 2)$.
- (c) $(n, \mu) = (2, 2)$. In this case, $\dim \mathfrak{g}(\mathfrak{m}) = \infty$. Furthermore, $\mathfrak{g}(\mathfrak{m})$ is isomorphic to $K(1)$ as a GLA.

8 Free pseudo-product fundamental graded Lie algebras

An FGLA $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ equipped with nonzero subspaces $\mathfrak{e}, \mathfrak{f}$ of \mathfrak{g}_{-1} is called a pseudo-product FGLA if the following conditions hold:

- (i) $\mathfrak{g}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$;
- (ii) $[\mathfrak{e}, \mathfrak{e}] = [\mathfrak{f}, \mathfrak{f}] = \{0\}$.

The pair $(\mathfrak{e}, \mathfrak{f})$ is called the pseudo-product structure of the pseudo-product FGLA $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$. We will also denote by the triplet $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ the pseudo-product FGLA $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ (resp. $\mathfrak{m}' = \bigoplus_{p < 0} \mathfrak{g}'_p$) be a pseudo-product FGLA with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$ (resp. $(\mathfrak{e}', \mathfrak{f}')$). We say that two pseudo-product FGLAs $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ and $(\mathfrak{m}'; \mathfrak{e}', \mathfrak{f}')$ are isomorphic if there exists a GLA isomorphism φ of \mathfrak{m} onto \mathfrak{m}' such that $\varphi(\mathfrak{e}) = \mathfrak{e}'$ and $\varphi(\mathfrak{f}) = \mathfrak{f}'$.

Proposition 8.1. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a pseudo-product FGLA of the μ -th kind with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$. If \mathfrak{m} is a free FGLA of type (n, μ) , then $n = 2$.

Proof. Let (e_1, \dots, e_m) (resp. (f_1, \dots, f_l)) be a basis of \mathfrak{e} (resp. \mathfrak{f}). Since $[\mathfrak{e}, \mathfrak{f}] = \mathfrak{g}_{-2}$, the space \mathfrak{g}_{-2} is generated by $\{[e_i, f_j] : i = 1, \dots, m, j = 1, \dots, l\}$ as a vector space, so $\dim \mathfrak{g}_{-2} \leq ml$. On the other hand, since \mathfrak{m} is a free FGLA,

$$\dim \mathfrak{g}_{-2} = \dim b(\mathfrak{g}_{-1}, \mu)_{-2} = \dim \Lambda^2(\mathfrak{g}_{-1}) = \frac{(m+l)(m+l-1)}{2},$$

so $ml \geq \frac{(m+l)(m+l-1)}{2}$. From this fact it follows that $m = l = 1$. ■

Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a pseudo-product FGLA of the μ -th kind with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$, where $\mu \geq 2$. \mathfrak{m} is called a free pseudo-product FGLA of type (m, n, μ) if the following conditions hold:

- (i) $\dim \mathfrak{e} = m$ and $\dim \mathfrak{f} = n$;
- (ii) Let $\mathfrak{m}' = \bigoplus_{p < 0} \mathfrak{g}'_p$ be a pseudo-product FGLA of the μ -th kind with pseudo-product structure $(\mathfrak{e}', \mathfrak{f}')$ and let φ be a surjective linear mapping of \mathfrak{g}_{-1} onto \mathfrak{g}'_{-1} such that $\varphi(\mathfrak{e}) \subset \mathfrak{e}'$ and $\varphi(\mathfrak{f}) \subset \mathfrak{f}'$. Then φ can be extended uniquely to a GLA epimorphism of \mathfrak{m} onto \mathfrak{m}' .

Proposition 8.2. Let m, n and μ be positive integers such that $\mu \geq 2$.

- (1) *There exists a unique free pseudo-product FGLA of type (m, n, μ) up to isomorphism.*
- (2) *Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free pseudo-product FGLA of type (m, n, μ) with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$. We denote by $\text{Der}(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})_0$ the Lie algebra of all the derivations of \mathfrak{m} preserving the gradation of \mathfrak{m} , \mathfrak{e} and \mathfrak{f} . Then the mapping $\Phi : \text{Der}(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})_0 \ni D \mapsto (D|_{\mathfrak{e}}, D|_{\mathfrak{f}}) \in \mathfrak{gl}(\mathfrak{e}) \times \mathfrak{gl}(\mathfrak{f})$ is a Lie algebra isomorphism.*

Proof. (1) The uniqueness of a free pseudo-product FGLA of type (m, n, μ) follows from the definition. Let V be an $(m + n)$ -dimensional vector space and let $\mathfrak{e}, \mathfrak{f}$ be subspaces of V such that $V = \mathfrak{e} \oplus \mathfrak{f}$, $\dim \mathfrak{e} = m$ and $\dim \mathfrak{f} = n$. Let $\mathfrak{a} = \bigoplus_{p < 0} \mathfrak{a}_p$ be the graded ideal of $b(V, \mu)$ generated by $[\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]$. We set $\mathfrak{m} = b(V, \mu)/\mathfrak{a}$, $\mathfrak{g}_p = b(V, \mu)_p/\mathfrak{a}_p$. Clearly $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ is a pseudo-product FGLA. We show that the factor algebra \mathfrak{m} is a free pseudo-product FGLA of type (m, n, μ) . First we prove that \mathfrak{m} is of the μ -th kind. Let $\mathfrak{n} = \bigoplus_{p < 0} \mathfrak{g}_p''$ be a free FGLA of type $(2, \mu)$ and let \mathfrak{e}'' and \mathfrak{f}'' be one-dimensional subspaces of \mathfrak{g}_{-1}'' such that $\mathfrak{g}_{-1}'' = \mathfrak{e}'' \oplus \mathfrak{f}''$. Let φ_1 be an injective linear mapping of \mathfrak{g}_{-1}'' into V such that $\varphi_1(\mathfrak{e}'') \subset \mathfrak{e}$ and $\varphi_1(\mathfrak{f}'') \subset \mathfrak{f}$. Let φ_2 be a linear mapping of V into \mathfrak{g}_{-1}'' such that $\varphi_2 \circ \varphi_1 = 1_{\mathfrak{g}_{-1}''}$, $\varphi_2(\mathfrak{e}) = \mathfrak{e}''$ and $\varphi_2(\mathfrak{f}) = \mathfrak{f}''$. There exists a homomorphism $L(\varphi_1)$ (resp. $L(\varphi_2)$) of \mathfrak{n} (resp. $b(V, \mu)$) into $b(V, \mu)$ (resp. \mathfrak{n}) such that $L(\varphi_1)|_{\mathfrak{g}_{-1}''} = \varphi_1$ (resp. $L(\varphi_2)|_V = \varphi_2$). Since $L(\varphi_2)([\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]) = \{0\}$, $L(\varphi_2)$ induces a homomorphism $\hat{L}(\varphi_2)$ of \mathfrak{m} into \mathfrak{n} such that $L(\varphi_2) = \hat{L}(\varphi_2) \circ \pi$, where π is the natural projection of $b(V, \mu)$ onto \mathfrak{m} . Since

$$1_{\mathfrak{n}} = L(\varphi_2) \circ L(\varphi_1) = \hat{L}(\varphi_2) \circ \pi \circ L(\varphi_1),$$

$\pi \circ L(\varphi_1)$ is a monomorphism of \mathfrak{n} into \mathfrak{m} , so $\mathfrak{g}_{-\mu} \neq \{0\}$. Thus \mathfrak{m} is of the μ -th kind. Let $\mathfrak{m}' = \bigoplus_{p < 0} \mathfrak{g}_p'$ be a pseudo-product FGLA of the μ -th kind with pseudo-product structure $(\mathfrak{e}', \mathfrak{f}')$ and let ϕ be a surjective linear mapping of $b(V, \mu)_{-1}$ onto \mathfrak{g}_{-1}' such that $\phi(\mathfrak{e}) \subset \mathfrak{e}'$ and $\phi(\mathfrak{f}) \subset \mathfrak{f}'$. By the definition of a free FGLA, there exists a GLA epimorphism $L(\phi)$ of $b(V, \mu)$ onto \mathfrak{m}' such that $L(\phi)|_{b(V, \mu)_{-1}} = \phi$. Since $L(\phi)([\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]) \subset [\mathfrak{e}', \mathfrak{e}'] + [\mathfrak{f}', \mathfrak{f}'] = \{0\}$, we see that $L(\phi)(\mathfrak{a}) = \{0\}$, so the epimorphism $L(\phi)$ induces a GLA epimorphism $\hat{L}(\phi)$ of \mathfrak{m} onto \mathfrak{m}' such that $\hat{L}(\phi)|_{\mathfrak{g}_{-1}} = \phi$.

(2) We may prove the fact that the mapping Φ is surjective. Let ϕ be an endomorphism of \mathfrak{g}_{-1} such that $\phi(\mathfrak{e}) \subset \mathfrak{e}$ and $\phi(\mathfrak{f}) \subset \mathfrak{f}$. By Proposition 2.1 (2), there exists a $D \in \text{Der}(b(V, \mu))_0$ such that $D|_{b(V, \mu)_{-1}} = \phi$. Since $D([\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]) \subset [\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]$, D induces a derivation \hat{D} of \mathfrak{m} such that $\hat{D}|_{\mathfrak{g}_{-1}} = \phi$. ■

Remark 8.1. Let m, n, m', n' and μ be positive integers with $\mu \geq 2$, and let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ (resp. $\mathfrak{m}' = \bigoplus_{p < 0} \mathfrak{g}_p'$) be a free pseudo-product FGLA of type (m, n, μ) (resp. (m', n', μ)) with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$ (resp. $(\mathfrak{e}', \mathfrak{f}')$). Furthermore let φ be a linear mapping of \mathfrak{g}_{-1} into \mathfrak{g}_{-1}' such that $\varphi(\mathfrak{e}) \subset \mathfrak{e}'$ and $\varphi(\mathfrak{f}) \subset \mathfrak{f}'$.

- (1) From the proof of Proposition 8.2, there exists a unique GLA homomorphism $\hat{L}(\varphi)$ of \mathfrak{m} into \mathfrak{m}' such that $\hat{L}(\varphi)|_{\mathfrak{g}_{-1}} = \varphi$. If φ is injective, then $\hat{L}(\varphi)$ is a monomorphism.
- (2) Assume that $m = n = 1$ and φ is injective. Then $\hat{L}(\varphi)(\mathfrak{m})$ is a graded subalgebra of \mathfrak{m}' isomorphic to a free FGLA of type $(2, \mu)$. From this result, the subalgebra of \mathfrak{m}' generated by a nonzero element X of \mathfrak{e}' and a nonzero element Y of \mathfrak{f}' is a free FGLA of type $(2, \mu)$.

Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a pseudo-product FGLA of the μ -th kind with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$. We denote by \mathfrak{g}_0 the Lie algebra of all the derivations of \mathfrak{m} preserving the gradation

of \mathfrak{m} , \mathfrak{e} and \mathfrak{f} :

$$\mathfrak{g}_0 = \{D \in \text{Der}(\mathfrak{g})_0 : D(\mathfrak{e}) \subset \mathfrak{e}, D(\mathfrak{f}) \subset \mathfrak{f}\}.$$

The prolongation $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of $(\mathfrak{m}, \mathfrak{g}_0)$ is called the prolongation of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$.

A transitive GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is called a pseudo-product GLA if there are given nonzero subspaces \mathfrak{e} and \mathfrak{f} of \mathfrak{g}_{-1} satisfying the following conditions:

- (i) The negative part \mathfrak{g}_- is a pseudo-product FGLA with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$;
- (ii) $[\mathfrak{g}_0, \mathfrak{e}] \subset \mathfrak{e}$ and $[\mathfrak{g}_0, \mathfrak{f}] \subset \mathfrak{f}$.

The pair $(\mathfrak{e}, \mathfrak{f})$ is called the pseudo-product structure of the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$. If the \mathfrak{g}_0 -modules \mathfrak{e} and \mathfrak{f} are irreducible, then the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is said to be of irreducible type.

The following lemma is due to N. Tanaka (cf. [9]). Here we give a proof for the convenience of the readers.

Lemma 8.1. *Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a pseudo-product GLA of depth μ with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$.*

- (1) *If \mathfrak{g}_- is non-degenerate, then \mathfrak{g} is finite-dimensional.*
- (2) *If $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is of irreducible type and $\mu \geq 2$, then \mathfrak{g} is finite-dimensional.*

Proof. (1) The proof is essentially due to the proof of [11, Corollary 3 to Theorem 11.1]. For $p \in \mathbb{Z}$, we set $\mathfrak{h}_p = \{X \in \mathfrak{g}_p : [X, \mathfrak{g}_{\leq -2}] = \{0\}\}$. We define $I \in \mathfrak{gl}(\mathfrak{g}_{-1})$ as follows: $I(x) = -\sqrt{-1}x$ for $x \in \mathfrak{e}$, $I(x) = \sqrt{-1}x$ for $x \in \mathfrak{f}$. Then $I^2 = -1$, $I([a, x]) = [a, I(x)]$ and $[I(x), I(y)] = [x, y]$ for $a \in \mathfrak{g}_0$ and $x, y \in \mathfrak{g}_{-1}$. We put $\langle x, y \rangle = [I(x), y]$ for $x, y \in \mathfrak{g}_{-1}$. Then $\langle x, y \rangle = \langle y, x \rangle$, and for $x \in \mathfrak{g}_{-1}$, $\langle x, \mathfrak{g}_{-1} \rangle = \{0\}$ implies $x = 0$, since \mathfrak{g}_- is non-degenerate. Also $\langle [a, x], y \rangle + \langle x, [a, y] \rangle = 0$ and $[[b, x], y] = [[b, y], x]$ for $a \in \mathfrak{h}_0$, $b \in \mathfrak{h}_1$ and $x, y \in \mathfrak{g}_{-1}$. Then, for $b \in \mathfrak{h}_1$, $x, y, z \in \mathfrak{g}_{-1}$, we have $\langle [[b, x], y], z \rangle = -\langle y, [[b, x], z] \rangle = -\langle y, [[b, z], x] \rangle = \langle [[b, z], y], x \rangle = \langle [[b, y], z], x \rangle = -\langle z, [[b, y], x] \rangle = -\langle [[b, x], y], z \rangle$, so $\langle [[b, x], y], z \rangle = 0$. By transitivity of \mathfrak{g} , $\mathfrak{h}_1 = \{0\}$. Therefore by [11, Corollary 1 to Theorem 11.1], \mathfrak{g} is finite-dimensional.

(2) We may assume that $\mathfrak{g}_1 \neq \{0\}$. By [16, Lemma 2.4], the \mathfrak{g}_0 -modules $\mathfrak{e}, \mathfrak{f}$ are not isomorphic to each other. We put $\mathfrak{d} = \{X \in \mathfrak{g}_{-1} : [X, \mathfrak{g}_{-1}] = \{0\}\}$; then \mathfrak{d} is a \mathfrak{g}_0 -submodule of \mathfrak{g}_{-1} . Hence $\mathfrak{d} = \{0\}$, $\mathfrak{d} = \mathfrak{e}$, $\mathfrak{d} = \mathfrak{f}$ or $\mathfrak{d} = \mathfrak{g}_{-1}$. If $\mathfrak{d} \neq \{0\}$, then $\mathfrak{g}_{-2} = [\mathfrak{e}, \mathfrak{f}] = \{0\}$, which is a contradiction. Thus \mathfrak{g}_- is non-degenerate. By (1), \mathfrak{g} is finite-dimensional. \blacksquare

The prolongation of a pseudo-product FGLA becomes a pseudo-product GLA. By Proposition 8.2 (2), the prolongation of a free pseudo-product FGLA is a pseudo-product GLA of irreducible type. By Lemma 8.1 (2), the prolongation of a free pseudo-product FGLA is finite-dimensional.

Proposition 8.3. *Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free pseudo-product FGLA of type (m, n, μ) with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$ and let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$.*

- (1) \mathfrak{g}_0 is isomorphic to $\mathfrak{gl}(\mathfrak{e}) \oplus \mathfrak{gl}(\mathfrak{f})$ as a Lie algebra.
- (2) \mathfrak{g}_{-2} is isomorphic to $\mathfrak{e} \otimes \mathfrak{f}$ as a \mathfrak{g}_0 -module. In particular, $\dim \mathfrak{g}_{-2} = mn$.

- (3) \mathfrak{g}_{-3} is isomorphic to $S^2(\mathfrak{e}) \otimes \mathfrak{f} \oplus S^2(\mathfrak{f}) \otimes \mathfrak{e}$ as a \mathfrak{g}_0 -module. In particular, $\dim \mathfrak{g}_{-3} = \frac{mn(m+n+2)}{2}$.

Proof. (1) This follows from Proposition 8.2 (2).

(2) Let $\mathfrak{a} = \bigoplus_{p < 0} \mathfrak{a}_p$ be the graded ideal of $b(\mathfrak{g}_{-1}, \mu)$ generated by $[\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]$. By the construction of $b(\mathfrak{g}_{-1}, \mu)_{-2}$, \mathfrak{a}_{-2} is isomorphic to $\Lambda^2(\mathfrak{e}) \oplus \Lambda^2(\mathfrak{f})$, so $\mathfrak{g}_{-2} = b(\mathfrak{g}_{-1}, \mu)_{-2}/\mathfrak{a}_{-2}$ is isomorphic to $\mathfrak{e} \otimes \mathfrak{f}$.

(3) By the construction of $b(\mathfrak{g}_{-1}, \mu)_{-3}$, $b(\mathfrak{g}_{-1}, \mu)_{-3}$ is isomorphic to

$$(\mathfrak{e} \oplus \mathfrak{f}) \otimes \Lambda^2(\mathfrak{e} \oplus \mathfrak{f}) / \Lambda^3(\mathfrak{e} \oplus \mathfrak{f}) \cong (\mathfrak{e} \otimes \mathfrak{e} \otimes \mathfrak{f}) \oplus (\mathfrak{e} \otimes \mathfrak{f} \otimes \mathfrak{f}).$$

Moreover, \mathfrak{a}_{-3} is isomorphic to

$$(\mathfrak{e} \oplus \mathfrak{f}) \otimes \Lambda^2(\mathfrak{e}) \oplus (\mathfrak{e} \oplus \mathfrak{f}) \otimes \Lambda^2(\mathfrak{f}) / \Lambda^3(\mathfrak{e} \oplus \mathfrak{f}) \cong \mathfrak{e} \otimes \Lambda^2(\mathfrak{e}) \oplus \mathfrak{f} \otimes \Lambda^2(\mathfrak{f}).$$

Hence $\mathfrak{g}_{-3} = b(\mathfrak{g}_{-1}, \mu)_{-3}/\mathfrak{a}_{-3}$ is isomorphic to

$$(\mathfrak{e} \otimes \mathfrak{e} \otimes \mathfrak{f}) / \Lambda^2(\mathfrak{e}) \otimes \mathfrak{f} \oplus (\mathfrak{e} \otimes \mathfrak{f} \otimes \mathfrak{f}) / \mathfrak{e} \otimes \Lambda^2(\mathfrak{f}) \cong S^2(\mathfrak{e}) \otimes \mathfrak{f} \oplus S^2(\mathfrak{f}) \otimes \mathfrak{e}.$$

This completes the proof. ■

Proposition 8.4. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a pseudo-product FGLA of the μ -th kind with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$, where $\mu \geq 2$. We denote by \mathfrak{c} the centralizer of \mathfrak{g}_{-2} in \mathfrak{g}_{-1} . Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$. Assume that \mathfrak{g}_0 is isomorphic to $\mathfrak{gl}(\mathfrak{e}) \oplus \mathfrak{gl}(\mathfrak{f})$ as a Lie algebra.

- (1) If $\mu = 2$, then $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free pseudo-product FGLA.
- (2) If $\mu \geq 3$ and $\mathfrak{c} \neq \{0\}$, then $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ is not a free pseudo-product FGLA.
- (3) If $\mu = 3$ and $\mathfrak{c} = \{0\}$, then $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ is a free pseudo-product FGLA.

Proof. Let $\check{\mathfrak{m}} = \bigoplus_{p < 0} \check{\mathfrak{g}}_p$ be the free pseudo-product FGLA of type (m, n, μ) with pseudo-product structure $(\check{\mathfrak{e}}, \check{\mathfrak{f}})$ such that $\check{\mathfrak{g}}_{-1} = \mathfrak{g}_{-1}$, $\check{\mathfrak{e}} = \mathfrak{e}$ and $\check{\mathfrak{f}} = \mathfrak{f}$. Let $\check{\mathfrak{g}} = \bigoplus_{p \in \mathbb{Z}} \check{\mathfrak{g}}_p$ be the prolongation of $(\check{\mathfrak{m}}; \check{\mathfrak{e}}, \check{\mathfrak{f}})$. There exists a GLA epimorphism φ of $\check{\mathfrak{m}}$ onto \mathfrak{m} such that the restriction $\varphi|_{\check{\mathfrak{g}}_{-1}}$ is the identity mapping. Since the mapping $\check{\mathfrak{g}}_0 \ni D \mapsto (D|\mathfrak{e}, D|\mathfrak{f}) \in \mathfrak{gl}(\mathfrak{e}) \times \mathfrak{gl}(\mathfrak{f})$ is an isomorphism, φ can be extended to be a homomorphism $\check{\varphi}$ of $\bigoplus_{p \leq 0} \check{\mathfrak{g}}_p$ onto $\bigoplus_{p \leq 0} \mathfrak{g}_p$. Let \mathfrak{a} be the kernel of $\check{\varphi}$; then \mathfrak{a} is a graded ideal of $\bigoplus_{p \leq 0} \check{\mathfrak{g}}_p$. We set $\mathfrak{a}_p = \mathfrak{a} \cap \check{\mathfrak{g}}_p$; then $\mathfrak{a} = \bigoplus_{p \leq 0} \mathfrak{a}_p$. Since the restriction of $\check{\varphi}$ to $\check{\mathfrak{g}}_{-1} \oplus \check{\mathfrak{g}}_0$ is injective, $\mathfrak{a}_p = \{0\}$ for $p \geq -1$. Also each \mathfrak{a}_p is a $\check{\mathfrak{g}}_0$ -submodule of $\check{\mathfrak{g}}_p$. Since the $\check{\mathfrak{g}}_0$ -module $\check{\mathfrak{g}}_{-2}$ is irreducible (Proposition 8.3 (2)), $\varphi|_{\check{\mathfrak{g}}_{-2}}$ is injective. If $\mu = 2$, then φ is an isomorphism. This proves the assertion (1). Now we assume that $\mu \geq 3$. Then

$$\check{\mathfrak{g}}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{f}] \oplus [[\mathfrak{e}, \mathfrak{f}], \mathfrak{e}].$$

Since $\check{\mathfrak{g}}_0$ -modules $[[\mathfrak{e}, \mathfrak{f}], \mathfrak{f}]$ and $[[\mathfrak{e}, \mathfrak{f}], \mathfrak{e}]$ are irreducible and not isomorphic to each other (Proposition 8.3 (3)), one of the following cases occurs: (i) $\mathfrak{a}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{f}]$; (ii) $\mathfrak{a}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{e}]$; (iii) $\mathfrak{a}_{-3} = \{0\}$. If $\mathfrak{a}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{f}]$ (resp. $\mathfrak{a}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{e}]$), then $\mathfrak{c} = \mathfrak{f}$ (resp. $\mathfrak{c} = \mathfrak{e}$). Also since \mathfrak{g}_0 -modules $\mathfrak{e}, \mathfrak{f}$ are irreducible and not isomorphic to each other, one of the following cases occurs: (i) $\mathfrak{c} = \mathfrak{e}$; (ii) $\mathfrak{c} = \mathfrak{f}$; (iii) $\mathfrak{c} = \{0\}$. If $\mathfrak{c} = \mathfrak{e}$ (resp. $\mathfrak{c} = \mathfrak{f}$), then $\mathfrak{a}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{e}]$ (resp. $\mathfrak{a}_{-3} = [[\mathfrak{e}, \mathfrak{f}], \mathfrak{f}]$). In this case, φ is not injective. Hence $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ is not free. If $\mathfrak{c} = \{0\}$, then $\mathfrak{a}_{-3} = \{0\}$. Hence $\varphi|_{\check{\mathfrak{g}}_{-3}}$ is an isomorphism. In particular, if $\mu = 3$, then $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ is free. ■

Example 8.1. Let V and W be finite-dimensional vector spaces and $k \geq 1$. We set

$$\begin{aligned}\mathfrak{C}^k(V, W) &= \bigoplus_{p=-k-1}^{-1} \mathfrak{C}^k(V, W)_p, \\ \mathfrak{C}^k(V, W)_p &= W \otimes S^{k+p+1}(V^*), \quad -k-1 \leq p \leq -2, \\ \mathfrak{C}^k(V, W)_{-1} &= V \oplus (W \otimes S^k(V^*)).\end{aligned}$$

The bracket operation of $\mathfrak{C}^k(V, W)$ is defined as follows:

$$\begin{aligned}[W, V] &= \{0\}, \quad [V, V] = \{0\}, \quad [W \otimes S^r(V^*), W \otimes S^s(V^*)] = \{0\}, \\ [w \otimes s_r, v] &= w \otimes (v \lrcorner s_r) \quad \text{for } v \in V, w \in W, s_r \in S^r(V^*).\end{aligned}$$

Equipped with this bracket operation, $\mathfrak{C}^k(V, W)$ becomes a pseudo-product FGLA of the $(k+1)$ -th kind with pseudo-product structure $(V, W \otimes S^k(V^*))$, which is called *the contact algebra of order k of bidegree (n, m)* , where $n = \dim V$ and $m = \dim W$ (cf. [14, p. 133]). We assume that $\mathfrak{C}^k(V, W)$ is a free pseudo-product FGLA. Since

$$\dim \mathfrak{C}^k(V, W)_{-2} = m \binom{n+k-2}{k-1}, \quad \dim V \dim(W \otimes S^k(V^*)) = nm \binom{n+k-1}{k},$$

we get $n = 1$. Since $W \otimes S^k(V^*)$ is contained in the centralizer of $\mathfrak{C}^k(V, W)_{-2}$ in $\mathfrak{C}^k(V, W)_{-1}$, we get $k = 1$. Thus we obtain that $\mathfrak{C}^k(V, W)$ is a free pseudo-product FGLA if and only if $k = 1$, $n = 1$.

Example 8.2. Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a finite-dimensional simple GLA of type $(A_{m+n}, \{\alpha_m, \alpha_{m+1}\})$.

We set $\mathfrak{e} = \mathfrak{g}_{-1}^{(m)}$, $\mathfrak{f} = \mathfrak{g}_{-1}^{(m+1)}$. Then $(\mathfrak{g}_-; \mathfrak{e}, \mathfrak{f})$ is a pseudo-product FGLA. Since $\dim \mathfrak{e} = m$, $\dim \mathfrak{f} = n$ and $\dim \mathfrak{g}_{-2} = mn$, the pseudo-product FGLA $(\mathfrak{g}_-; \mathfrak{e}, \mathfrak{f})$ is a free pseudo-product FGLA of type $(m, n, 2)$ (Proposition 8.3 (2)). Also $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is the prolongation of \mathfrak{g}_- except

for the following cases (see [15]):

- (1) $m = n = 1$. In this case, the prolongation of \mathfrak{g}_- is isomorphic to $K(1)$.
- (2) $m = 1$ or $n = 1$ and $l = \max\{m, n\} \geq 2$. In this case, the prolongation of \mathfrak{g}_- is isomorphic to $W(l+1; \mathbf{s})$, where $\mathbf{s} = (1, 2, \dots, 2)$.

Example 8.3. Let V and W be finite-dimensional vector spaces such that $\dim V = m \geq 1$ and $\dim W = n \geq 1$. We set

$$\begin{aligned}\mathfrak{g}_{-1} &= V \oplus W, \quad \mathfrak{g}_{-2} = V \otimes W, \\ \mathfrak{g}_{-3} &= V \otimes S^2(W) \oplus S^2(V) \otimes W, \quad \mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3}.\end{aligned}$$

The bracket operation of \mathfrak{m} is defined as follows:

$$\begin{aligned}[\mathfrak{g}_{-3}, \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}] &= [\mathfrak{g}_{-2}, \mathfrak{g}_{-2}] = \{0\}, \quad [V, V] = [W, W] = \{0\}, \\ [v, w] &= -[w, v] = v \otimes w, \quad [v, v' \otimes w] = -[v' \otimes w, v] = v \odot v' \otimes w, \\ [v \otimes w, w'] &= -[w', v \otimes w] = v \otimes w \odot w',\end{aligned}$$

where $v, v' \in V$ and $w, w' \in W$. Equipped with this bracket operation, \mathfrak{m} becomes a free pseudo-product FGLA of type $(m, n, 3)$ with pseudo-product structure (V, W) (Proposition 8.3).

Theorem 8.1. *Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free pseudo-product FGLA of type (m, n, μ) with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$ over \mathbb{C} . Furthermore let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ (resp. $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$) be the prolongation of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ (resp. \mathfrak{m}).*

- (1) *Assume that $\dim \mathfrak{g}(\mathfrak{m}) = \infty$. Then $m = 1$ or $n = 1$, and $\mu = 2$. Furthermore $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is isomorphic to a finite-dimensional simple GLA of type $(A_{l+1}, \{\alpha_1, \alpha_2\})$, where $l = \max\{m, n\}$. If $l = 1$, then $\mathfrak{g}(\mathfrak{m})$ is isomorphic to $K(1)$. If $l \geq 2$, then $\mathfrak{g}(\mathfrak{m})$ is isomorphic to $W(l+1; \mathbf{s})$, where $\mathbf{s} = (1, 2, \dots, 2)$.*
- (2) *If $\mathfrak{g}_1 \neq \{0\}$, then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is a finite-dimensional simple GLA of type $(A_{m+n}, \{\alpha_m, \alpha_{m+1}\})$.*

Proof. (1) For $p \geq -1$, we put $\mathfrak{h}_p = \{X \in \mathfrak{g}(\mathfrak{m})_p : [X, \mathfrak{g}_{\leq -2}] = \{0\}\}$. Assume that $\dim \mathfrak{g}(\mathfrak{m}) = \infty$ and $\mu \geq 3$. By Proposition 8.4 (2), $\mathfrak{h}_{-1} = \{0\}$. Since $[\mathfrak{h}_0, \mathfrak{g}_{-1}] \subset \mathfrak{h}_{-1} = \{0\}$, we see that $\mathfrak{h}_0 = \{0\}$. By [11, Corollary 1 to Theorem 11.1], we obtain that $\dim \mathfrak{g}(\mathfrak{m}) < \infty$, which is a contradiction. Thus we see that $\mu = 2$ if $\dim \mathfrak{g}(\mathfrak{m}) = \infty$. The remaining assertion follows from Example 8.2.

(2) Assume that $\mathfrak{g}_1 \neq \{0\}$ and $\mu \geq 3$. By transitivity of \mathfrak{g} , $[\mathfrak{g}_1, \mathfrak{e}] \neq \{0\}$ or $[\mathfrak{g}_1, \mathfrak{f}] \neq \{0\}$. We may assume that $[\mathfrak{g}_1, \mathfrak{e}] \neq \{0\}$. Then there exists an irreducible component \mathfrak{g}'_1 of the \mathfrak{g}_0 -module \mathfrak{g}_1 such that $[\mathfrak{g}'_1, \mathfrak{e}] \neq \{0\}$ and $[\mathfrak{g}'_1, \mathfrak{f}] = \{0\}$. The subalgebra $\mathfrak{e} \oplus [\mathfrak{e}, \mathfrak{g}'_1] \oplus \mathfrak{g}'_1$ is a simple GLA of the first kind. Since \mathfrak{g}_0 is isomorphic to $\mathfrak{gl}(\mathfrak{e}) \oplus \mathfrak{gl}(\mathfrak{f})$, $\mathfrak{e} \oplus [\mathfrak{e}, \mathfrak{g}'_1] \oplus \mathfrak{g}'_1$ is of type $(A_m, \{\alpha_1\})$. Let D be a nonzero element of \mathfrak{g}'_1 . There exist $\lambda \in \mathfrak{e}^*$ and $\eta \in \mathfrak{f}^*$ such that

$$[[D, Z], U] = \lambda(U)Z + \lambda(Z)U, \quad [[D, Z], W] = \eta(Z)W,$$

where $Z, U \in \mathfrak{e}$ and $W \in \mathfrak{f}$ (cf. [12, p. 4]). Let X (resp. Y) be a nonzero element of \mathfrak{e} (resp. \mathfrak{f}). Since the subalgebra generated by X, Y is a free FGLA of type $(2, \mu)$ (Remark 8.1 (2)),

$$\begin{aligned} \operatorname{ad}(X)^\mu(Y) &= 0, & \operatorname{ad}(X)^{\mu-1}(Y) &\neq 0, \\ \operatorname{ad}(Y) \operatorname{ad}(X)^{\mu-1}(Y) &= 0, & \operatorname{ad}(Y) \operatorname{ad}(X)^{\mu-2}(Y) &\neq 0 \end{aligned}$$

(Lemma 2.1). By induction on μ , we see that

$$\begin{aligned} 0 &= \operatorname{ad}(D) \operatorname{ad}(X)^\mu(Y) = (\mu(\mu-1)\lambda(X) + \mu\eta(X)) \operatorname{ad}(X)^{\mu-1}(Y), \\ 0 &= \operatorname{ad}(D) \operatorname{ad}(Y) \operatorname{ad}(X)^{\mu-1}(Y) \\ &= ((\mu-1)(\mu-2)\lambda(X) + (\mu-1)\eta(X)) \operatorname{ad}(Y) \operatorname{ad}(X)^{\mu-2}(Y). \end{aligned}$$

Since

$$\det \begin{bmatrix} \mu(\mu-1) & \mu \\ (\mu-1)(\mu-2) & \mu-1 \end{bmatrix} = \mu(\mu-1) \neq 0,$$

we see that $\lambda(X) = \eta(X) = 0$, which is a contradiction. Thus we obtain that $\mu = 2$ if $\dim \mathfrak{g}_1 \neq \{0\}$. From Example 8.2, it follows that $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is a simple GLA of type $(A_{m+n}, \{\alpha_m, \alpha_{m+1}\})$ if $\dim \mathfrak{g}_1 \neq \{0\}$. ■

9 Automorphism groups of the prolongations of free pseudo-product fundamental graded Lie algebras

For a GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ we denote by $\operatorname{Aut}(\mathfrak{g})_0$ the group of all the automorphisms of \mathfrak{g} preserving the gradation of \mathfrak{g} :

$$\operatorname{Aut}(\mathfrak{g})_0 = \{\varphi \in \operatorname{Aut}(\mathfrak{g}) : \varphi(\mathfrak{g}_p) = \mathfrak{g}_p \text{ for all } p \in \mathbb{Z}\}.$$

Proposition 9.1. *Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be an FGLA and let $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ be the prolongation of \mathfrak{m} . The mapping $\Phi : \text{Aut}(\mathfrak{g}(\mathfrak{m}))_0 \ni \phi \mapsto \phi|_{\mathfrak{m}} \in \text{Aut}(\mathfrak{m})_0$ is an isomorphism.*

Proof. It is clear that Φ is a group homomorphism. We prove that Φ is injective. Let ϕ be an element of $\text{Ker } \Phi$. Assume that $\phi(X) = X$ for all $X \in \mathfrak{g}(\mathfrak{m})_p$ ($p < k$). For $X \in \mathfrak{g}(\mathfrak{m})_k$, $Y \in \mathfrak{g}_{-1}$,

$$[\phi(X) - X, Y] = \phi([X, Y]) - [X, Y].$$

Since $[X, Y] \in \mathfrak{g}(\mathfrak{m})_{k-1}$, we have $[\phi(X) - X, Y] = 0$. By transitivity, $\phi(X) = X$. By induction, we have proved ϕ to be the identity mapping. Hence Φ is a monomorphism.

We prove that Φ is surjective. Let $\varphi \in \text{Aut}(\mathfrak{m})_0$. We construct the mapping $\varphi_p : \mathfrak{g}(\mathfrak{m})_p \rightarrow \mathfrak{g}(\mathfrak{m})_p$ inductively as follows: First for $X \in \mathfrak{g}(\mathfrak{m})_0$, we set $\varphi_0(X) = \varphi X \varphi^{-1}$. Then for $Y, Z \in \mathfrak{m}$

$$\varphi_0(X)([Y, Z]) = [\varphi(X(\varphi^{-1}(Y))), Z] + [Y, \varphi(X(\varphi^{-1}(Z)))],$$

so $\varphi_0(X) \in \mathfrak{g}(\mathfrak{m})_0$. Furthermore we can prove easily that $[\varphi_0(X), \varphi_p(Y)] = \varphi_p([X, Y])$ for $X \in \mathfrak{g}_0$ and $Y \in \mathfrak{g}_p$ ($p \leq 0$). Here for $p < 0$ we set $\varphi_p = \varphi|_{\mathfrak{g}(\mathfrak{m})_p}$. Assume that we have defined linear isomorphisms φ_p of $\mathfrak{g}(\mathfrak{m})_p$ onto itself ($0 \leq p < k$) such that

$$\varphi_{r+s}([X, Y]) = [\varphi_r(X), \varphi_s(Y)]$$

for $X \in \mathfrak{g}(\mathfrak{m})_r$, $Y \in \mathfrak{g}(\mathfrak{m})_s$ ($r + s < k$, $r < k$, $s < k$). For $X \in \mathfrak{g}(\mathfrak{m})_k$ we define $\varphi_k(X) \in \text{Hom}(\mathfrak{m}, \bigoplus_{p \leq k-1} \mathfrak{g}(\mathfrak{m})_p)_k$ as follows:

$$\varphi_k(X)(Y) = \varphi_{k+s}([X, \varphi^{-1}(Y)]), \quad Y \in \mathfrak{g}_s, \quad s < 0.$$

For $Y \in \mathfrak{g}_s$, $Z \in \mathfrak{g}_t$ ($s, t < 0$),

$$\begin{aligned} \varphi_k(X)([Y, Z]) &= \varphi_{k+t+s}([X, \varphi^{-1}([Y, Z])]) \\ &= \varphi_{k+s+t}([X, \varphi^{-1}(Y)], \varphi^{-1}(Z)] + [\varphi^{-1}(Y), [X, \varphi^{-1}(Z)])] \\ &= [\varphi_{k+s}([X, \varphi^{-1}(Y)]), Z] + [Y, \varphi_{k+t}([X, \varphi^{-1}(Z)])] \\ &= [\varphi_k(X)(Y), Z] + [Y, \varphi_k(X)(Z)], \end{aligned}$$

so $\varphi_k(X) \in \mathfrak{g}(\mathfrak{m})_k$. Next we prove that for $X \in \mathfrak{g}_p$, $Y \in \mathfrak{g}_q$ ($p + q = k$, $0 \leq p \leq k$, $0 \leq q \leq k$),

$$\varphi_k([X, Y]) = [\varphi_p(X), \varphi_q(Y)].$$

For $Z \in \mathfrak{g}_s$ ($s < 0$),

$$\begin{aligned} [[\varphi_p(X), \varphi_q(Y)], Z] &= [\varphi_p(X), [\varphi_q(Y), Z]] - [\varphi_q(Y), [\varphi_p(X), Z]] \\ &= \varphi_{p+q+s}([X, [Y, \varphi^{-1}(Z)]] - [Y, [X, \varphi^{-1}(Z)]]) \\ &= \varphi_{p+q+s}([X, Y], \varphi^{-1}(Z)) = [\varphi_k([X, Y]), Z]. \end{aligned}$$

By transitivity, we see that $\varphi_k([X, Y]) = [\varphi_p(X), \varphi_q(Y)]$. We define a mapping $\check{\varphi}$ of $\mathfrak{g}(\mathfrak{m})$ into itself as follows:

$$\check{\varphi}(X) = \begin{cases} \varphi(X), & X \in \mathfrak{m}, \\ \varphi_k(X), & k \geq 0, X \in \mathfrak{g}(\mathfrak{m})_k. \end{cases}$$

From the above results and the definition of φ_k ($k \geq 0$), we see that $\check{\varphi}$ is a GLA homomorphism.

Assume that φ_{k-1} ($k \geq 0$) is a linear isomorphism. For $X \in \mathfrak{g}(\mathfrak{m})_k$, if $\varphi_k(X) = 0$, then $0 = [\varphi_k(X), Y] = \varphi_{k-1}([X, \varphi^{-1}(Y)])$ for all $Y \in \mathfrak{g}_{-1}$. By transitivity, we see that $X = 0$, so φ_k is a linear isomorphism. Therefore $\check{\varphi}$ is an automorphism of $\mathfrak{g}(\mathfrak{m})$. \blacksquare

Theorem 9.1. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free FGLA over \mathbb{C} , and let $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ be the prolongation of \mathfrak{m} . The mapping $\Phi : \text{Aut}(\mathfrak{g}(\mathfrak{m}))_0 \ni \phi \mapsto \phi|_{\mathfrak{g}_{-1}} \in GL(\mathfrak{g}_{-1})$ is an isomorphism.

Proof. We may assume that \mathfrak{m} is a universal FGLA $b(\mathfrak{g}_{-1}, \mu)$ of the μ -th kind. By Corollary 1 to Proposition 3.2 of [11], the mapping $\text{Aut}(\mathfrak{m})_0 \ni a \mapsto a|_{\mathfrak{g}_{-1}} \in GL(\mathfrak{g}_{-1})$ is an isomorphism. By Proposition 9.1, we see that the mapping $\Phi : \text{Aut}(\mathfrak{g}(\mathfrak{m}))_0 \ni \phi \mapsto \phi|_{\mathfrak{g}_{-1}} \in GL(\mathfrak{g}_{-1})$ is an isomorphism. ■

For a pseudo-product GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$, we denote by $\text{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0$ the group of all the automorphisms of \mathfrak{g} preserving the gradation of \mathfrak{g} , \mathfrak{e} and \mathfrak{f} :

$$\text{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0 = \{\varphi \in \text{Aut}(\mathfrak{g})_0 : \varphi(\mathfrak{e}) = \mathfrak{e}, \varphi(\mathfrak{f}) = \mathfrak{f}\}.$$

Theorem 9.2. Let $\mathfrak{m} = \bigoplus_{p < 0} \mathfrak{g}_p$ be a free pseudo-product FGLA of type (m, n, μ) with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$ over \mathbb{C} , and let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$. The mapping $\Phi : \text{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0 \ni \phi \mapsto (\phi|_{\mathfrak{e}}, \phi|_{\mathfrak{f}}) \in GL(\mathfrak{e}) \times GL(\mathfrak{f})$ is an isomorphism. Furthermore if $\dim \mathfrak{e} \neq \dim \mathfrak{f}$, then $\text{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0 = \text{Aut}(\mathfrak{g})_0$.

Proof. Clearly Φ is a monomorphism. We show that Φ is surjective. Let (ϕ_1, ϕ_2) be an element of $GL(\mathfrak{e}) \times GL(\mathfrak{f})$. We set $\phi = \phi_1 \oplus \phi_2 \in GL(\mathfrak{g}_{-1})$. By Corollary 1 to Proposition 3.2 of [11], there exists an element $\varphi_1 \in \text{Aut}(b(\mathfrak{g}_{-1}, \mu))_0$ such that $\varphi_1|_{\mathfrak{g}_{-1}} = \phi$. Since $\varphi_1([\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]) = [\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]$, φ_1 induces an element $\varphi_2 \in \text{Aut}(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})_0$ such that $\varphi_2|_{\mathfrak{g}_{-1}} = \phi$. By Proposition 9.1, there exists $\varphi_3 \in \text{Aut}(\mathfrak{g}(\mathfrak{m}))_0$ such that $\varphi_3|_{\mathfrak{m}} = \varphi_2$. We prove that $\varphi_3(\mathfrak{g}) = \mathfrak{g}$. For $X_0 \in \mathfrak{g}_0$ and $Y \in \mathfrak{e}$, we see that $[\varphi_3(X_0), Y] = \varphi_3([X_0, \varphi_3^{-1}(Y)]) \in \varphi_3(\mathfrak{e}) = \mathfrak{e}$, so $\varphi_3(X_0)(\mathfrak{e}) \subset \mathfrak{e}$. Similarly we get $\varphi_3(X_0)(\mathfrak{f}) \subset \mathfrak{f}$. Thus we obtain that $\varphi_3(\mathfrak{g}_0) = \mathfrak{g}_0$. Now we assume that $\varphi_i(\mathfrak{g}_i) = \mathfrak{g}_i$ for $0 \leq i \leq k$. Then for $X_{k+1} \in \mathfrak{g}_{k+1}$ and $Y \in \mathfrak{g}_p$ ($p < 0$), we see that $[\varphi_3(X_{k+1}), Y] = \varphi_3([X_{k+1}, \varphi_3^{-1}(Y)]) \in \varphi_3(\mathfrak{g}_{p+k+1}) = \mathfrak{g}_{p+k+1}$, so $\varphi_3(\mathfrak{g}_{k+1}) \subset \mathfrak{g}_{k+1}$. Hence $\varphi_3(\mathfrak{g}) = \mathfrak{g}$ and Φ is surjective. Now we assume that $\dim \mathfrak{e} \neq \dim \mathfrak{f}$. Let $\varphi \in \text{Aut}(\mathfrak{g})_0$. Since \mathfrak{g}_0 -modules \mathfrak{e} and \mathfrak{f} are not isomorphic to each other, we see that (i) $\varphi(\mathfrak{e}) = \mathfrak{e}$, $\varphi(\mathfrak{f}) = \mathfrak{f}$ or (ii) $\varphi(\mathfrak{e}) = \mathfrak{f}$, $\varphi(\mathfrak{f}) = \mathfrak{e}$. According to the assumption $\dim \mathfrak{e} \neq \dim \mathfrak{f}$, we get $\varphi(\mathfrak{e}) = \mathfrak{e}$, $\varphi(\mathfrak{f}) = \mathfrak{f}$, so $\varphi \in \text{Aut}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})_0$. ■

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